ANALYSIS OF THE SACK PROBLEM WITH DETAILED INCLUSION OF INTERATOMIC COHESIVE FORCES

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Elsewhere [1] we considered the Griffiths problem with detailed inclusion of the interatomic cohesive forces acting between the edges of a crack. The cohesive forces are introduced into the boundary conditions and as a result the problem reduces to a nonlinear integrodifferential equation. In a similar formulation in this paper we consider an axisymmetric problem of the tension of an elastic space weakened by a flat crack in the plane. The problem is reduced to one of solving a nonlinear integrodifferential equation, which is done by methods of regular and spliced asymptotic expansions. Using one of the asymptotic solutions, we also obtained a numerical solution of the integrodifferential equation under study. The parameters of the critical state of the crack are determined from the conditions of smooth closing of the crack edges.

1. Supopse that an elastic space with a regular atomic lattice contains a round crack of radius a in the plane z = 0. The crack is in an open state under tensile forces $\sigma_z = p =$ const applied at infinity. When the normal interatomic distance b is exceeded cohesive forces arise between the layers of atoms; the strength σ_z of these forces can be taken in the form [1]

$$\sigma_z = 2\theta\varepsilon g(\varepsilon/d) \quad (\theta = G(1-\nu)^{-1}). \tag{1.1}$$

Here G is the shear modulus; ν is Poisson's ratio; $\varepsilon = \Delta b/b$; $b + \Delta b$ is the distance between layers of atoms; and $d = \delta/b$ is the relative distance between layers of atoms at which the cohesive forces reach a maximum equal to σ_p, the theoretical strength of the solid. The function g(x) decreases monotonically no slower than $x^{-\alpha}$ ($\alpha > 2$) and satisfies the conditions

$$g(0) = 1, \quad g(\infty) = 0, \quad g(1) + g'(1) = 0.$$

On the basis of (1.1) we find the effective surface energy density of the medium [2]

$$\gamma = \frac{b}{2}\int_d^\infty \sigma_z d\varepsilon = \frac{\delta\sigma_p I}{2g(1)} \quad \left(I = \int_1^\infty xg(x)\,dx\right). \tag{1.2}$$

Suppose that $\Gamma(r) = 2u_z(r, +0)$ is the crack opening (u_z is a component of the displacement vector). We assume that the crack begins where the distance between layers of atoms becomes equal to $b + \delta$, which means that $\Gamma(a) = 0$ and $\sigma_z(a, \pm 0) = \sigma_p$ on the crack contour. Inclusion of the interatomic cohesive forces results in the following boundary conditions:

$$z = 0, \quad \tau_{rz} = 0 \quad (0 \leqslant r < \infty), \quad u_z = 0 \quad (a < r < \infty),$$
$$\sigma_z = \frac{\sigma_p}{g(1)}\left(1 + \frac{\Gamma}{\delta}\right) g\left(1 + \frac{\Gamma}{\delta}\right) \quad (0 \leqslant r \leqslant a).$$

At infinity $\sigma_z = p$. We disregard the fact that by (1.1) the problem is physically nonlinear in the vicinity of the crack contour and assume that the equations of the linear theory of elasticity are valid everywhere outside the crack. The application of the integral Hankel transform to the problem obtained for the function $\Gamma(r)$ reduces the problem to one of solving the nonlinear integrodifferential equation

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$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\int_0^1 x\Gamma(x)\,dx\int_0^\infty J_0(ur)\,J_0(ux)\,du = \frac{1}{\lambda}\,[f(\Gamma) - p], \tag{1.3}$$

$$\Gamma(1) = 0, \quad 0 \leqslant r \leqslant 1, \quad f(\Gamma) = (1+\Gamma)\,g(1+\Gamma)/g(1),$$

where $J_0(z)$ is a Bessel function:

$$\Gamma_* = \Gamma/\delta; \quad p_* = p/\sigma_p; \quad r_* = r/a; \quad \lambda = b\,[4ag(1)]^{-1}.$$

The asterisk is omitted in (1.3) and below. The critical load p is determined from the condition [3]

$$\Gamma'(1) = 0. \tag{1.4}$$

Inverting the operator on the left side of (1.3), we reduce (1.3) to the form

$$\Gamma(r) = 2p\,(\pi\lambda)^{-1}\sqrt{1-r^2} - \Lambda f(\Gamma)/\lambda, \quad \Gamma(1) = 0, \quad 0 \leqslant r \leqslant 1, \tag{1.5}$$

$$\Lambda\omega = \frac{2}{\pi}\int_r^1 \frac{d\xi}{\sqrt{\xi^2 - r^2}}\int_0^\xi \frac{x\omega(x)}{\sqrt{\xi^2 - x^2}}\,dx.$$

Condition (1.4) with allowance for (1.5) leads to the following representation of p:

$$p = \int_0^1 \frac{xf(\Gamma)}{\sqrt{1-x^2}}\,dx. \tag{1.6}$$

Considering that $\max|f(x)| = f(0) = 1$ for $x \in [0, \infty)$, we can show that p is less than 1. It is sufficient for this purpose to estimate the integral (1.6),

$$p \leqslant \int_0^1 \frac{x\,|f(\Gamma)|}{\sqrt{1-x^2}}\,dx \leqslant \max\nolimits_\Gamma |f(\Gamma)| \int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} = 1.$$

We note that the trivial solution $\Gamma = 0$ of Eq. (1.5), (1.6) occurs for $p = 1$.

2. The solution of (1.5), (1.6) for small values of λ is constructed as asymptotic expansions

$$\Gamma(r) = \lambda\,[\Gamma_0(r) + \lambda\Gamma_1(r) + \lambda^2\Gamma_2(r) + O(\lambda^3)]; \tag{2.1}$$

$$p = 1 - \lambda^2[A_0 + \lambda A_1 + \lambda^2 A_2 + O(\lambda^3)]. \tag{2.2}$$

Coefficients A_n ($n = 0, 1, 2$) can be determined from the condition that follows from (1.4):

$$\Gamma_n'(1) = 0 \quad (n = 0, 1, 2). \tag{2.3}$$

Introducing (2.1) and (2.2) into (1.5) and (1.6) and equating the expressions for identical powers of λ, we obtain integral equations from which we successively find $\Gamma_0(r)$, $\Gamma_1(r)$, and $\Gamma_2(r)$:

$$\Gamma_n(r) - 2B_0\Lambda(\Gamma_0\Gamma_n) = \Lambda\psi_n - 2A_n\sqrt{1-r^2}/\pi, \tag{2.4}$$

$$\psi_n(r) = [B_0\Gamma_1^2(r) - 3B_1\Gamma_0^2(r)\,\Gamma_1(r)]\,\delta_{n2} - B_n\Gamma_0^{n+2}(r) \quad (n = 0, 1, 2),$$

$$B_0 = 1 - \frac{g''(1)}{2g(1)}, \quad B_1 = \frac{g''(1) + (1/3)\,g'''(1)}{2g(1)}, \quad B_2 = \frac{g'''(1) + (1/4)\,g^{IV}(1)}{6g(1)}.$$

Here δ_{mn} is the Kronecker symbol. Condition (2.3) leads to the following representations of the coefficients A_n:

$$A_n = \int_0^1 [2B_0\Gamma_0(x)\,\Gamma_n(x) + \psi_n(x)]\,\frac{x\,dx}{\sqrt{1-x^2}} \quad (n = 0, 1, 2). \tag{2.5}$$

We note that the integral equation (2.4) is nonlinear for n = 0 and linear for n = 1, 2. To eliminate the expansion factor B_0 when n = 0 we write $\Gamma_0(r)$ as

$$\Gamma_0(r) = C\varphi(r)/B_0.$$

Then from (2.4), (2.5) we have an integral equation for determining $\phi(r)$:

$$\varphi(r) = C\Lambda\varphi^2 - 2\sqrt{1-r^2}/\pi, \qquad C = \left[\int_0^1 \frac{x\varphi^2(x)\,dx}{\sqrt{1-x^2}}\right]^{-1}. \tag{2.6}$$

An approximate solution of Eq. (2.6) can be obtained by the method of successive approximations using the scheme

$$\varphi_{m+1}(r) = \varphi_0(r) + C_m\Lambda\varphi_m^2 \quad (m = 0, 1, \ldots, M), \tag{2.7}$$

$$\varphi_0(r) = -\frac{2}{\pi}\sqrt{1-r^2}, \qquad C_m = \left[\int_0^1 \frac{x\varphi_m^2(x)\,dx}{\sqrt{1-x^2}}\right]^{-1}.$$

The functions $\phi_m(r)$ can be found in explicit form for each value of m when scheme (2.7) is realized.

It is desirable for the values n = 1, 2 to recast Eqs. (2.4), (2.5) into a form that does not contain A_n:

$$\Gamma_n(r) - 2B_0\Lambda_1(\Gamma_0\Gamma_n) = \Lambda_1\psi_n \quad (n = 1, 2), \tag{2.8}$$

$$\Lambda_1\omega = \Lambda\omega - \frac{2}{\pi}\sqrt{1-r^2}\int_0^1 \frac{x\omega(x)}{\sqrt{1-x^2}}\,dx.$$

The solution of Eq. (2.8) can be obtained by the Bubnov–Galerkin method. With this method we look for the solution of Eq. (2.8) in the form

$$\Gamma_n(r) = \frac{2}{\pi}\sqrt{1-r^2}\sum_{i=0}^{\infty} X_{ni}U_{2i}(r) \quad (n = 1, 2), \tag{2.9}$$

where $U_m(r)$ are Chebyshev polynomials of the second kind. The application of the Bubnov–Galerkin procedure to Eq. (2.8) with allowance for (2.9) reduces it to the following systems of linear algebraic equations in coefficients X_{ni} (n = 1, 2):

$$X_{nj} - \frac{4B_0}{\pi}\sum_{i=0}^{\infty} X_{ni}H_{ji} = D_{nj} \quad (j = 0, 1, \ldots), \tag{2.10}$$

$$H_{ji} = 2\int_0^1\int_0^1 \sqrt{\frac{1-t^2}{1-x^2}}\, t\Gamma_0(t)\,P_j(2y^2-1)\,U_{2i}(t)\,dx\,dy -$$

$$- \delta_{j0}\int_0^1 x\Gamma_0(x)\,U_{2i}(x)\,dx \quad (t = xy),$$

$$D_{nj} = 2\int_0^1\int_0^1 xy\psi_n(xy)\,P_j(2y^2-1)\,\frac{dx\,dy}{\sqrt{1-x^2}} - \delta_{j0}\int_0^1 \frac{x\psi_n(x)}{\sqrt{1-x^2}}\,dx.$$

Here $P_j(x)$ are Legendre polynomials. Since condition (2.3) was satisfied by the choice of constants A_n, the coefficients X_{ni} determined from (2.10) should satisfy the relation

$$\sum_{i=0}^{\infty}(2i+1)X_{ni} = 0 \quad (n = 1, 2).$$

Direct calculations from the above formulas were made for

$$g(x) = \exp(-x). \tag{2.11}$$

These calculations gave $B_0 = 1/2$, $B_1 = 1/3$, $B_2 = -1/8$, $A_0 = 2.14$, $A_1 = 20.5$, $A_2 = 153$. Systems (2.10) for n = 1 and n = 2 were solved by the reduction method.

3. The solution found in Sec. 2 corresponds to relatively long cracks with a relatively small opening. Let us now consider the case of a relatively large opening. For this purpose we introduce the notation

$$\mu = \lambda/p, \quad \Gamma^1 = \mu\Gamma.$$

As a result Eq. (1.3) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\int_0^1 x\Gamma(x)\,dx\int_0^\infty J_0(ur)J_0(ux)\,du = \frac{\mu}{\lambda}f\left(\frac{\Gamma}{\mu}\right) - 1 \quad (0 \leqslant r \leqslant 1); \tag{3.1}$$

$$\Gamma(1) = 0, \quad \Gamma'(1) = 0. \tag{3.2}$$

The subscript 1 is omitted from Γ in (3.1), (3.2), and henceforth. Inverting the operator in (3.1), (3.2), we have

$$\Gamma(r) = \frac{2}{\pi}\sqrt{1 - r^2} - \frac{2\mu}{\pi\lambda}\int_r^1 \frac{d\xi}{\sqrt{\xi^2 - r^2}}\int_0^\xi \frac{xf(\Gamma/\mu)}{\sqrt{\xi^2 - x^2}}\,dx \quad (0 \leqslant r \leqslant 1). \tag{3.3}$$

This equation is equivalent to (3.1), (3.2) upon satisfaction of the relation

$$\int_0^1 \frac{xf(\Gamma/\mu)}{\sqrt{1 - x^2}}\,dx = \frac{\lambda}{\mu}, \tag{3.4}$$

which follows from the second condition (3.2). To solve Eqs. (3.3), (3.4) for small values of μ we use the method of spliced asymptotic expansions [4]. The external (penetrating) solution outside the contour of the crack for $\mu \ll 1$ and with allowance for the properties of the function g(x) can be obtained from (3.3). It has the form

$$\Gamma_0(r) = 2\sqrt{1 - r^2}/\pi.$$

We now consider the ε-neighborhood of the point r = 1. We introduce the notation:

$$\rho = (1 - r)/\varepsilon, \quad s = (1 - \xi)/\varepsilon, \quad t = (1 - x)/\varepsilon. \tag{3.5}$$

As the boundary of the ε-neighborhood is approached the external solution with (3.5) taken into account assumes the form

$$\Gamma_0(r) = 2\sqrt{2\varepsilon\rho}/\pi.$$

For splicing, therefore, we look for the internal solution in the form

$$\Gamma(r) = \sqrt{\varepsilon}\,q(\rho) + o(\sqrt{\varepsilon}). \tag{3.6}$$

Introducing (3.6) into (3.3) and going over to the internal variable, we find

$$q(\rho) = \frac{2}{\pi}\sqrt{2\rho} - \frac{\varkappa}{\pi}\int_0^\rho \frac{ds}{\sqrt{\rho - s}}\int_s^\infty \frac{f(q)}{\sqrt{t - s}}\,dt \quad (0 \leqslant \rho < \infty), \quad \sqrt{\varepsilon} = \mu, \quad \varkappa = \mu^2/\lambda.$$

Then, changing the order of integration and calculating the internal integral, we finally obtain

$$q(\rho) = \frac{2}{\pi}\sqrt{2\rho} - \frac{\varkappa}{\pi}\int_0^\infty f(q)\ln\frac{\sqrt{\zeta} + \sqrt{\rho}}{|\sqrt{\zeta} - \sqrt{\rho}|}\,d\zeta \quad (0 \leqslant \rho < \infty). \tag{3.7}$$

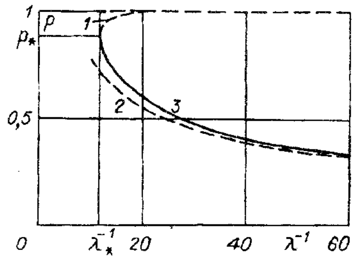

Fig. 1

Similarly, condition (3.4) becomes

$$\varkappa J = 1 \qquad \left(J = \int_0^\infty \frac{f(q)\,dt}{\sqrt{2t}}\right). \tag{3.8}$$

From (3.8) we find the critical force

$$p = \sqrt{\lambda J}. \tag{3.9}$$

We note that in the given problem on the basis of the Griffiths energy principle for determining the critical force Sack obtained the formula [2]

$$p = \sqrt{\pi\theta\gamma/a}, \tag{3.10}$$

which we write in the dimensionless quantities used here as

$$p = \sqrt{\pi\lambda I/g(1)}. \tag{3.11}$$

For the case (2.11) we obtained $p = \sqrt{2.00\pi\lambda}$ and $p = \sqrt{2\pi\lambda}$ from formulas (3.7)-(3.9) and (1.2), (3.11), respectively. The integral equation (3.7) was solved by the method of successive approximations, with $q_0(\rho) = 2\sqrt{2\rho}/\pi$ taken for the zeroth approximation. In the numerical integration the logarithmic singularity in the integrand in (3.7) was eliminated by using integral 4.339 of [5].

4. An approximate solution of the integral equation (1.5), (1.6) can be found by the method of successive approximations. As the zeroth approximation for the specific value of λ we must take one of the asymptotic solutions obtained above. Then, changing λ in fairly small steps, as the zeroth approximation we must take the solution of Eq. (1.5), (1.6) found for the previous value of λ. Figure 1 shows the values of the limiting load p calculated from (2.2), (3.9), and (1.6) (curves 1-3, respectively) for $g(x) = \exp(-x)$. The lower branch of curve 3 for $\lambda_*^{-1} < \lambda^{-1}$ and $p < p^*$ ($\lambda_* = 0.0865$, $p_* = 0.890$) since in an open crack in the first place a stress-strain state corresponding to this branch is reached as the p load increases monotonically. For $\lambda^{-1} < \lambda_*^{-1}$ (or $a < 7.85b$) the limiting point p is 1, which corresponds to fracture after the theoretical breaking point. For $\lambda^{-1} \geq 55$ the values of p obtained from the Sack formula (3.11) and from (1.6) differ by less than 3%.

LITERATURE CITED

1. V. M. Aleksandrov and I. I. Kudish, "Asymptotic methods in the Griffiths problem," Prikl. Mat. Mekh., 53, No. 4 (1989).
2. V. V. Panasyuk, Limit Equilibrium of Cracked Brittle Solids [in Russian], Naukova Dumka, Kiev (1968).
3. Yu. P. Zheltov and S. A. Khristianovich, "On the hydraulic fracturing of an oil-bearing bed," Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk, No. 5 (1955).
4. Ali-Hasan, Perturbation Methods, Wiley-Interscience, New York (1973).
5. I. S. Gradshteyn and I. M. Ryshik, Tables of Integrals, Series & Products, Academic Press, New York (1966).